

# Precision Gauge Unification from Extra Yukawa Couplings

Ivan Donkin and Arthur Hebecker

*Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 19, D-69120  
Heidelberg, Germany*

(a.hebecker@thphys.uni-heidelberg.de , i.donkin@thphys.uni-heidelberg.de)

## Abstract

We investigate the impact of extra vector-like GUT multiplets on the predicted value of the strong coupling. We find in particular that Yukawa couplings between such extra multiplets and the MSSM Higgs doublets can resolve the familiar two-loop discrepancy between the SUSY GUT prediction and the measured value of  $\alpha_3$ . Our analysis highlights the advantages of the holomorphic scheme, where the perturbative running of gauge couplings is saturated at one loop and further corrections are conveniently described in terms of wavefunction renormalization factors. If the gauge couplings as well as the extra Yukawas are of  $\mathcal{O}(1)$  at the unification scale, the relevant two-loop correction can be obtained analytically. However, the effect persists also in the weakly-coupled domain, where possible non-perturbative corrections at the GUT scale are under better control.

# 1 Introduction

The consistency of low-energy data with supersymmetric gauge coupling unification [1] is one of the strongest reasons to expect the discovery of supersymmetry at the LHC. Moreover, gauge coupling unification is very well-motivated in heterotic string compactifications (see [2] for some of the recent developments) as well as in F-theory [3]. Thus, if supersymmetry is discovered, the SUSY GUT framework will provide one of the most direct ways to access the fundamental high-scale theory (see [4] for a recent phenomenological study).

The SUSY GUT prediction for the strong coupling is, however, not perfect. In fact, using two-loop  $\beta$  functions and identifying the effective SUSY breaking scale with  $m_Z$ , the prediction misses the measured value of  $\alpha_3(m_Z)$  by many standard deviations. Possible resolutions of this discrepancy include corrections due to an unusual SUSY spectrum (for a recent analysis see [5]) or unexpectedly large GUT thresholds.

In the present paper, we focus on a different possibility: In addition to the usual MSSM spectrum, we allow for extra chiral supermultiplets in complete vector-like GUT representations. Such states appear in many different contexts and are well-motivated theoretically (e.g. as the “messengers” of gauge mediation). While these extra multiplets do not affect the one-loop prediction for  $\alpha_3$ , the induced two-loop effect is significant. Unfortunately, it further enhances the familiar problems of the MSSM two-loop prediction [6].

However, with extra multiplets naturally come extra Yukawas. We focus on Yukawa couplings of the extra multiplets with the MSSM Higgs doublets (in line with the general structure of the MSSM and with the natural extension of R-parity to these multiplets). As a simple example one may think of a fourth vector-like generation. We also note that, following [7–9], a similar class of models has recently been considered in [10–12] as a possible solution to the little hierarchy problem. Furthermore, extra multiplets may also improve the little hierarchy problem in the context of [13]. We take all of this as an important extra motivation for our scenario.

Our results show that the new Yukawa interactions induce a significant shift in the strong coupling towards its correct experimental value. Moreover, we find that if the extra Yukawas are relatively large at the GUT scale, we end up with an almost perfect prediction for  $\alpha_3$ .

The paper is organized as follows: We start in Sect. 2 with a two-loop analysis of MSSM gauge unification in the holomorphic scheme. While gauge couplings run only at one loop, the  $Z$  factors of chiral multiplets receive corrections at all loop orders. However, to achieve two-loop precision for the  $\alpha_3$  prediction, it is sufficient work with one-loop  $Z$  factors. Their effect on the  $\alpha_3$  prediction comes from the transition to the canonical scheme (via the vector and Konishi anomalies), which we perform at the electroweak scale. In this approach, the two-loop correction to the MSSM prediction for  $\alpha_3$  arises from a sum of terms  $\sim \ln Z_f$ , where  $f$  runs over all flavors, including in particular the two Higgs doublets. It becomes apparent that a significant enhancement of the Higgs  $Z$

factors can provide the desired shift in the  $\alpha_3$  prediction.

Extra multiplets are introduced in Sect. 3.1. Their  $Z$  factors are not important since these fields are integrated out above the scale where the transition to the canonical scheme is performed. The detrimental effect of extra multiplets on gauge unification mentioned earlier arises solely through the increased value of the gauge couplings at high energies. The gauge couplings lead to decreasing  $Z$  factors as one runs from high to low energy scales, and this effect is enhanced in the presence of extra matter.

In Sect. 3.2 we introduce extra Yukawa couplings. We first consider an analytically calculable example with strong gauge coupling at the GUT scale. This can be realized introducing extra multiplets in the  $\mathbf{10} + \overline{\mathbf{10}}$  and  $\mathbf{5} + \overline{\mathbf{5}}$  of  $SU(5)$  with masses near the TeV scale. Assuming that the extra Yukawas of type  $\mathbf{10} \mathbf{10} H_U$  and  $\overline{\mathbf{10}} \overline{\mathbf{10}} H_D$  are also strong at the GUT scale and neglecting the small effect of the top Yukawa coupling, we solve this model analytically. The resulting shift in the  $\alpha_3$  prediction, which is due to enhanced Higgs  $Z$  factors, leads to nearly perfect agreement with the experimental value.

In Sect. 3.3 we extend our analysis to more general scenarios, keeping in particular the GUT-scale gauge coupling in the perturbative domain. We demonstrate that our promising initial results retain their validity in this setting as long as the extra Yukawa couplings at the unification scale are sufficiently large.

Scenarios with a larger number of extra Yukawa couplings are investigated in Sect. 3.4. We focus on two particular types of extension. First we analyze models in which further  $\mathbf{10} + \overline{\mathbf{10}}$  pairs with further extra Yukawas are introduced. We find that in this case the total effect on the Higgs  $Z$  factor can not be increased significantly. The second type of models we consider possess the same matter content as our minimal model from Sect. 3.3 (i.e. a  $\mathbf{10} + \overline{\mathbf{10}}$  plus a  $\mathbf{5} + \overline{\mathbf{5}}$  pair). However, we now also allow for couplings of type  $\overline{\mathbf{5}} \mathbf{10} H_D$  and  $\mathbf{5} \overline{\mathbf{10}} H_U$ . We find that in this case the two-loop prediction for the inverse coupling  $2\pi/\alpha_3$  is increased even further. In particular, we are able to reproduce the ‘optimal’ results from Sect. 3.3 under milder assumptions (i.e. for lower couplings at the GUT scale).

Low- and high-scale threshold corrections and, in particular, the critical issue of strong gauge couplings at the GUT scale [14–21] are discussed in Sect. 4. We emphasize that, due to the absence of higher-order perturbative corrections to the holomorphic couplings, precision unification is not compromised when allowing for relatively large values of the GUT coupling. However, once one reaches the actual strong-coupling regime, non-perturbative corrections can arise and affect the  $\alpha_3$  prediction. Thus, it appears to be safe to stay in a domain where terms that are exponentially suppressed by the inverse gauge coupling are negligible. This is sufficient for our purposes.

## 2 Conventional unification from a holomorphic perspective

### 2.1 Basic formulae with holomorphic thresholds

We find it convenient to work in a holomorphic scheme, where the perturbative running of gauge couplings is saturated at one loop [22–24]. Focussing for simplicity on SU(5) models with adjoint breaking, we have

$$\frac{2\pi}{\alpha_{h,i}(\mu)} = \frac{2\pi}{\alpha_h(M)} + b_i \ln \frac{M}{\mu} + b_i^{(3)} \ln \frac{M}{m_3^h} + b_i^{X,Y} \ln \frac{M}{m_{X,Y}^h} + b_i^\Phi \ln \frac{M}{m_\Phi^h}, \quad (1)$$

where  $\alpha_{h,i}(\mu)$ , with  $i = 1, 2, 3$ , are the holomorphic U(1), SU(2) and SU(3) gauge couplings of the MSSM at some low scale  $\mu$  and  $\alpha_h(M)$  is the unified holomorphic coupling at some high scale  $M$ . This equation is exact to all orders of perturbation theory provided that  $\mu$  lies above the supersymmetry breaking scale. While the first logarithm combines the effects of all fields that remain light below the GUT scale, the last three logarithms are associated with heavy fields. The various  $b_i$ 's are the appropriate one-loop  $\beta$ -function coefficients. They are labelled by ‘(3)’ for Higgs-triplet, ‘ $X, Y$ ’ for the massive vector multiplets of  $X, Y$  gauge bosons, and  $\Phi$  for those components of the GUT-breaking field  $\Phi$  that are not ‘eaten’ by the  $X, Y$  bosons. We emphasize that  $m_3^h$ ,  $m_{X,Y}^h$  and  $m_\Phi^h$  are holomorphic rather than physical mass scales. In other words, these are the mass parameters of the holomorphic Wilsonian action, in which kinetic terms are *not* canonically normalized [24].

At the moment, we may think of a situation where  $m_3^h \sim m_{X,Y}^h \sim m_\Phi^h \sim M_{\text{GUT}}^h$  and  $\mu \ll M_{\text{GUT}}^h \ll M$ . Furthermore, we identify the SUSY-breaking scale with  $m_Z$  for now, postponing a brief discussion of low-scale threshold effects to Sect. 4.

Thus, we can set  $\mu = m_Z$  and translate holomorphic to canonical gauge couplings using the well-known anomaly relation [22–24]<sup>1</sup>:

$$\frac{2\pi}{\alpha_{h,i}} = \frac{2\pi}{\alpha_i} + T(G_i) \ln g_i^2 + \sum_f T(R_i^f) \ln Z_f. \quad (2)$$

Here  $\alpha_i \equiv g_i^2/(4\pi)$  is the canonical gauge coupling,  $T(G_i) = C_2(G_i)$  is the Dynkin index or quadratic Casimir of the adjoint of the gauge group  $G_i$ , and  $T(R_i^f)$  is the Dynkin index of the representation  $R_i^f$  of the flavor  $f$  (with respect to  $G_i$ ). This gives rise to the ‘holomorphic master formula’<sup>2</sup>

$$\begin{aligned} \frac{2\pi}{\alpha_i(m_Z)} &= \frac{2\pi}{\alpha_h(M)} + b_i \ln \frac{M}{m_Z} + b_i^{(3)} \ln \frac{M}{m_3^h} + b_i^{X,Y} \ln \frac{M}{m_{X,Y}^h} + b_i^\Phi \ln \frac{M}{m_\Phi^h} \\ &\quad - T(G_i) \ln(g_i^2(m_Z)) - \sum_f T(R_i^f) \ln Z_f(m_Z). \end{aligned} \quad (3)$$

---

<sup>1</sup>At our level of accuracy, we can ignore corrections associated with the transition to the scheme-dependent (e.g. DRED) physical gauge coupling [25].

<sup>2</sup>i.e. the holomorphic version of what is called the ‘master formula’ in [16, 23]

Obviously, the choice of  $M$  in this formula is irrelevant for the prediction of  $\alpha_3$  from  $\alpha_1$  and  $\alpha_2$ . In particular, we can set  $M = m_{X,Y}^h$  (irrespective of the actual UV completion scale) and write

$$\frac{2\pi}{\alpha_i(m_Z)} = \frac{2\pi}{\alpha_h(m_{X,Y}^h)} + b_i \ln \frac{m_{X,Y}^h}{m_Z} - T(G_i) \ln(g_i^2(m_Z)) - \sum_f T(R_i^f) \ln Z_f(m_Z) + \Delta_{i,\text{GUT}}^h, \quad (4)$$

with (holomorphic) GUT threshold corrections

$$\Delta_{i,\text{GUT}}^h = b_i^{(3)} \ln \frac{m_{X,Y}^h}{m_3^h} + b_i^\Phi \ln \frac{m_{X,Y}^h}{m_\Phi^h}. \quad (5)$$

The above threshold corrections will be small if the two relevant mass ratios are  $\mathcal{O}(1)$ . This will be the case if the superpotential contains no parametrically small couplings. Indeed, staying strictly within the holomorphic scheme, not only  $m_\Phi^h$  and  $m_3^h$ , but also  $m_{X,Y}^h$  are determined purely by superpotential terms (in contrast to the physical masses of the  $X, Y$  multiplets, which include a prefactor  $g$  coming from the gauge kinetic term).

## 2.2 Compatibility with the conventional master formula

In order to establish the equivalence between Eq. (3) and the more conventional master formula of [23], we need to replace holomorphic by physical mass parameters:

$$m_{X,Y}^p = g m_{X,Y}^h Z_\Phi^{1/2}, \quad m_\Phi^p Z_\Phi = m_\Phi^h, \quad m_3^p Z_3 = m_3^h, \quad (6)$$

where  $g$ ,  $Z_\Phi$  and  $Z_3$  are the (GUT-scale) gauge coupling and  $Z$  factors<sup>3</sup> and the superscript ‘ $p$ ’ stands for ‘physical’. Using these relations together with Eq. (2), we rewrite Eq. (3) in terms of canonical gauge coupling and physical masses. Making also use of the identity  $b_i^{X,Y} - 2T(G_i) = -2T(\text{SU}(5))$ , which follows from  $b_i^{X,Y} = -2T(R_i^{X,Y})$  and  $T(R_i^{X,Y}) + T(G_i) = T(\text{SU}(5))$ , and choosing  $M = m_{X,Y}^p$ , we find

$$\frac{2\pi}{\alpha_i(m_Z)} = \frac{2\pi}{\alpha(m_{X,Y}^p)} + b_i \ln \frac{m_{X,Y}^p}{m_Z} - T(G_i) \ln \frac{\alpha_i(m_Z)}{\alpha(m_{X,Y}^p)} - \sum_f T(R_i^f) \ln \frac{Z_f(m_Z)}{Z_f(m_{X,Y}^p)} + \Delta_{i,\text{GUT}}^p, \quad (7)$$

with (physical) GUT threshold corrections

$$\Delta_{i,\text{GUT}}^p = b_i^{(3)} \ln \frac{m_{X,Y}^p}{m_3^p} + b_i^\Phi \ln \frac{m_{X,Y}^p}{m_\Phi^p}. \quad (8)$$

This is in agreement with [6, 16, 23]. It simply represents a slightly different parameterization of our ignorance of the high-scale input: In the ‘holomorphic master formula’

---

<sup>3</sup> Since they appear only in logarithms, it does not matter whether we use a holomorphic or a canonical gauge coupling and at which precise mass scale we evaluate all these quantities.

of the previous subsection, all the relevant non-holomorphic (and in this sense ‘unprotected’) input data were the high-scale boundary values of the Higgs  $Z$  factors. (Note that high-scale  $Z$  factors of complete  $SU(5)$  multiplets, such as MSSM matter, do not affect the  $\alpha_3$  prediction.) Now, by contrast, the very same ignorance is hidden in the value of the physical Higgs triplet mass.

## 2.3 Predicting $\alpha_3$

It will be convenient to rewrite Eqs. (4) and (5) as:

$$\frac{2\pi}{\alpha_i(m_Z)} = \frac{2\pi}{\alpha_h(m_{X,Y}^h)} + b_i \ln \frac{m_{X,Y}^h}{m_Z} + \Delta_i, \quad (9)$$

collecting all two-loop effects (the logs of gauge couplings and  $Z$ -factors as well as GUT thresholds) in the correction terms  $\Delta_i$ . Multiplying the first of these three equations by  $(b_2 - b_3)/(b_1 - b_2) = 5/7$ , the second by  $(b_3 - b_1)/(b_1 - b_2) = -12/7$ , and adding them to the third equation, one finds

$$\frac{2\pi}{\alpha_3} = \underbrace{\left[ -\frac{5}{7} \frac{2\pi}{\alpha_1} + \frac{12}{7} \frac{2\pi}{\alpha_2} \right]}_{\text{one-loop prediction}} + \underbrace{\left[ \frac{5}{7} \Delta_1 - \frac{12}{7} \Delta_2 + \Delta_3 \right]}_{\text{two-loop corrections}}. \quad (10)$$

Here we have suppressed the mass scale,  $m_Z$ , for brevity. Setting the holomorphic GUT thresholds to zero, the two-loop correction, i.e. the second bracket in Eq. (10), explicitly reads

$$\begin{aligned} \Delta_{2\text{-loop}} \left( \frac{2\pi}{\alpha_3} \right) &= \frac{24}{7} \ln g_2^2 - 3 \ln g_3^2 + \frac{27}{14} \ln Z_L - \frac{9}{7} \ln Z_E + \frac{9}{2} \ln Z_Q \\ &\quad - \frac{45}{14} \ln Z_U - \frac{27}{14} \ln Z_D + \frac{9}{14} \ln Z_{H_U} + \frac{9}{14} \ln Z_{H_D}. \end{aligned} \quad (11)$$

Given that we are only aiming at two-loop accuracy, it is consistent to evaluate all quantities entering the above expression with one-loop precision. In particular, we can use one-loop  $Z$ -factors and the experimental values for the gauge couplings (given by  $2\pi/\alpha_1 = 370.7$ ,  $2\pi/\alpha_2 = 185.8$ ,  $2\pi/\alpha_3 = 53.2$ ). The  $Z$  factors can be written as [23]

$$\begin{aligned} Z_{L,H_U,H_D} &= Z_{SU(2)} \times Z_{U(1)} \\ Z_Q &= Z_{SU(3)} \times Z_{SU(2)} \times Z_{U(1)}^{1/9} \\ Z_E &= Z_{U(1)}^4 \\ Z_U &= Z_{SU(3)} \times Z_{U(1)}^{16/9} \\ Z_D &= Z_{SU(3)} \times Z_{U(1)}^{4/9}. \end{aligned} \quad (12)$$

Introducing the shorthand notation  $Z_{U(1)} \equiv Z_1$ ,  $Z_{SU(2)} \equiv Z_2$ ,  $Z_{SU(3)} \equiv Z_3$ , the respective gauge group contributions read

$$Z_i = \left( \frac{\alpha_{\text{GUT}}}{\alpha_i(m_Z)} \right)^{-\frac{2C_{F,i}}{b_i}}. \quad (13)$$

Here  $C_{F,i}$  are the quadratic Casimir operators of the fundamental representations of  $U(1)$ ,  $SU(2)$  and  $SU(3)$ , whereas  $b_i$  are the respective one-loop  $\beta$ -function coefficients (cf. App. A). Furthermore,  $\alpha_{\text{GUT}}$  is the one-loop GUT coupling,  $2\pi/\alpha_{\text{GUT}} = 153$ . Equation (13) is valid under the assumption that all  $Z$ -factors are unity at  $M_{\text{GUT}}$ .

Combining all contributions, we find:

$$\begin{aligned} \frac{2\pi}{\alpha_3(m_Z)} &= \underbrace{53.7}_{1\text{-loop}} - \underbrace{4.08}_{\text{vector anomaly}} - \underbrace{0.64}_{Z_L} - \underbrace{5.51}_{Z_Q} + \underbrace{0.21}_{Z_E} + \underbrace{3.22}_{Z_U} + \underbrace{1.83}_{Z_D} - \underbrace{0.42}_{Z_{H_U}, Z_{H_D}} \\ &= 53.7 - 5.4 = 48.3, \end{aligned} \quad (14)$$

i.e., the two-loop corrections shift the one-loop prediction away from the experimental value of 53.2. As already advertised in the Introduction, this set of numbers suggests a simple way to cure the problem: It will be sufficient to introduce a significant enhancement of the Higgs  $Z$  factors at low energies. This will be realized in the following using extra multiplets with extra Yukawa couplings.

For completeness, we record the result which is obtained if the canonical (rather than the holomorphic) GUT threshold corrections are assumed to vanish. At the technical level, this corresponds simply to replacing the ‘vector anomaly’ contribution  $-4.08$  above with

$$\frac{24}{7} \ln \frac{\alpha_2(m_Z)}{\alpha_{\text{GUT}}} - 3 \ln \frac{\alpha_3(m_Z)}{\alpha_{\text{GUT}}} = -0.67 - 3.17 = -3.84. \quad (15)$$

The resulting prediction improves insignificantly.

The smallness of this change is due to the weak dependence of the  $\alpha_3$  prediction on the value of  $\alpha_{\text{GUT}}$  in Eq. (15). This is the result of the approximate cancellation  $24/7 - 3 = 3/7$ . It is equivalent to the statement that the splitting between  $m_{X,Y}$  and  $m_\Phi$  affects the  $\alpha_3$  prediction only very weakly, which reflects the similarity of the ratios of the  $b_i^\Phi$  and the ratios of the corresponding MSSM coefficients  $b_i$ .

Finally, we comment on the effect of MSSM Yukawa couplings which we have so far neglected. At moderate  $\tan\beta$ , the top Yukawa coupling gives the dominant correction (cf. App. B),

$$\Delta_{\text{top}} \left( \frac{2\pi}{\alpha_3} \right) = 0.32, \quad (16)$$

which is however far too small to cure the ‘two-loop’ problem.

## 3 Extra multiplets

### 3.1 Extra multiplets without Yukawa couplings

To preserve one-loop gauge coupling unification in the most straightforward way <sup>4</sup>, we restrict our attention to complete  $SU(5)$  multiplets. More specifically, we focus on models

---

<sup>4</sup>See [26] for an alternative point of view

with  $n_5$  pairs of  $\mathbf{5} + \overline{\mathbf{5}}$  and  $n_{10}$  pairs of  $\mathbf{10} + \overline{\mathbf{10}}$ . At one loop, this leads to a modification,  $b_i \rightarrow b'_i = b_i + n$ , of the MSSM  $\beta$  function coefficients, where  $n = n_5 + 3n_{10}$  is known as the ‘messenger index’. This, of course, does not affect the one-loop  $\alpha_3$  prediction.

The two-loop correction, given explicitly in Eq. (11), changes only because of the modified  $Z$  factors of the MSSM matter fields. We emphasize that Eq. (11) does *not* need to be supplemented with  $Z$  factors of the extra multiplets since these are assumed to decouple above  $m_Z$ .

The modified  $Z$  factors are obtained from Eqs. (12), as before, but now with the gauge group contributions (cf. Eq. (13))

$$Z_i(m_Z) = \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_i(m_Z)} \right)^{-\frac{2C_{F,i}}{b'_i}} = \left( \frac{\frac{2\pi}{\alpha_i(m_Z)}}{\frac{2\pi}{\alpha_{\text{GUT}}} - n \ln \frac{M_{\text{GUT}}}{m_Z}} \right)^{-\frac{2C_{F,i}}{b_i+n}}. \quad (17)$$

Here  $\hat{\alpha}_{\text{GUT}}$  is the one-loop GUT-coupling in the presence of extra multiplets. Note that the one-loop GUT scale,  $M_{\text{GUT}} = 2 \cdot 10^{16}$  GeV, is unaffected by the presence of the additional matter. For simplicity, we have at this stage neglected the necessary hierarchy between  $m_Z$  and the scale at which the extra multiplets decouple.

The two-loop corrections related to the different matter field  $Z$ -factors can be found in App. C. Here we only list the overall two-loop prediction for  $2\pi/\alpha_3$  for different values of the  $n$  parameter:

Table 1: Two-loop prediction for  $2\pi/\alpha_3$

$n$	Prediction for $2\pi/\alpha_3$
1	48.08
2	47.91
3	47.61
4	46.81
4.45	45.81

The value  $n = 4.45$  formally corresponds to  $\hat{\alpha}_{\text{GUT}} = 1$ . Of course, this has to be interpreted as  $n = 5$  together with an appropriately raised decoupling scale. We see that with increasing  $n$  the two-loop prediction for  $\alpha_3$  becomes systematically worse.

Of course, we could easily repeat the analysis using the assumption of vanishing canonical GUT-thresholds. In this case, the increased value of the GUT-coupling enters the result also via the analogue of Eq. (15), with  $\alpha_{\text{GUT}}$  replaced by  $\hat{\alpha}_{\text{GUT}}$ . For  $n = 4.45$ , this contribution changes from  $-3.84$  to  $-5.16$ , giving  $2\pi/\alpha_3 = 44.73$ .

### 3.2 Extra multiplets with Yukawa couplings - an example with strong GUT coupling

In the following we extend our analysis of Sect. 3.1 by allowing renormalizable couplings between the  $\text{SU}(5)$  matter and the MSSM particles. To this end let us assume that there



is at least one pair of  $\mathbf{10} + \overline{\mathbf{10}}$  vector-like matter. In analogy to the MSSM, we denote its field content by  $\mathbf{10} = (Q_e, U_e, E_e)$  and  $\overline{\mathbf{10}} = (\overline{Q}_e, \overline{U}_e, \overline{E}_e)$ , with an index ‘ $e$ ’ for ‘extra multiplet’. We can now introduce extra Yukawa couplings of the form:

$$W \supset \kappa Q_e U_e H_U + \overline{\kappa} \overline{Q}_e \overline{U}_e H_D, \quad (18)$$

These new interactions modify the prediction for  $\alpha_3$  solely through their effect on the Higgs wavefunction renormalization factor  $Z_H$  (where  $Z_H$  stands for either  $Z_{H_U}$  or  $Z_{H_D}$ ). We note that the couplings  $\kappa, \overline{\kappa}$  can be extended to full  $4 \times 4$  Yukawa matrices allowing for mixing between the three MSSM generations and the additional matter. However, these mixings have to be small due to FCNC constraints [11] (see also [27, 28]) and we neglect them.

In order to simplify the analysis in this section let us once again neglect any contributions arising from the MSSM Yukawa sector (based on our results from Sect. 2.3 and App. B we expect those contributions to be small). This allows us to treat  $\kappa$  and  $\overline{\kappa}$  on an equal footing. In particular we can set  $\kappa = \overline{\kappa}$  and effectively deal with a single Yukawa coupling (say  $\kappa$ ) and a single Higgs (calling this field  $H$ ).

The one-loop RGE for  $Z_H$  (with  $t = \ln \mu$ ) reads

$$2\pi \frac{d \ln Z_H}{dt} = -3\alpha_\kappa + \frac{3}{10}\alpha_1 + \frac{3}{2}\alpha_2, \quad (19)$$

where we have defined  $\alpha_\kappa = \kappa_c^2/(4\pi)$ , with  $\kappa_c^2 = \kappa^2/(Z_H Z_{Q_e} Z_{U_e})$  the *canonical* Yukawa coupling. It is clear that a large  $\kappa$  will drive  $Z_H$  to larger values at the electroweak scale, improving the  $\alpha_3$  prediction.

Equation (19) entails the factorization property

$$Z_H = \underbrace{Z_H^G}_{\text{gauge part}} \times \underbrace{Z_H^Y}_{\text{Yukawa part}}, \quad (20)$$

where  $Z_H^G$  and  $Z_H^Y$  represent the contributions from the gauge couplings  $\alpha_1, \alpha_2$  and the Yukawa coupling  $\alpha_\kappa$ . As before, the gauge part  $Z_H^G$  is determined by Eqs. (12) and (17). We thus focus on the one-loop running of  $Z_H^Y$ :

$$2\pi \frac{d \ln Z_H^Y}{dt} = -3\alpha_\kappa. \quad (21)$$

The corresponding one-loop RGE for  $\alpha_\kappa$  reads

$$2\pi \frac{d \ln \alpha_\kappa}{dt} = 6\alpha_\kappa - \frac{16}{3}\alpha_3 - 3\alpha_2 - \frac{13}{15}\alpha_1. \quad (22)$$

In the following we will analyze a model in which the value of the low-scale  $Z_H^Y$ -factor can be obtained in a completely analytical manner. To this end let us assume that both the extra Yukawa couplings as well as the gauge couplings begin their evolution at the strong-coupling point at the high scale:

$$\hat{\alpha}_{GUT} \sim \alpha_\kappa(M_{\text{GUT}}) \sim 1. \quad (23)$$

Formally this corresponds to  $n = 4.45$ . In this case the relations

$$\alpha_2 = \frac{b'_3}{b'_2} \alpha_3 \quad \alpha_1 = \frac{b'_3}{b'_1} \alpha_3 \quad (24)$$

will be approximately valid at all energies significantly below the GUT scale.<sup>5</sup> Equation (22) now takes the form

$$2\pi \frac{d \ln \alpha_\kappa}{dt} = 6\alpha_\kappa - \alpha_3 \left( \frac{16}{3} + \frac{3b'_3}{b'_2} + \frac{13b'_3}{15b'_1} \right). \quad (25)$$

Using the one-loop RGEs for the gauge couplings

$$\frac{d \ln \alpha_i^{-1}}{dt} = -b'_i \frac{\alpha_i}{2\pi}, \quad (26)$$

this can be further rewritten as

$$2\pi \frac{d \ln(\alpha_\kappa/\alpha_3)}{dt} = 6\alpha_\kappa - \alpha_3 \left( \frac{16}{3} + \frac{3b'_3}{b'_2} + \frac{13b'_3}{15b'_1} + b'_3 \right). \quad (27)$$

Eq. (27) has an infrared-stable fixed point of the Pendleton-Ross type [29] given by

$$\alpha_\kappa = 1.37\alpha_3 \quad \text{for} \quad n = 4.45. \quad (28)$$

As a result of fast initial evolution, the ratios of Yukawa and gauge couplings quickly reach the fixed-point regime, which is then maintained all the way down to the weak scale. From Eq. (21) we now have

$$\frac{d \ln Z_H^Y}{dt} = -3 \cdot 1.37 \frac{\alpha_3}{2\pi} = \frac{4.11}{b'_3} \frac{d \ln \alpha_3^{-1}}{dt} \quad (29)$$

and

$$\ln Z_H^Y(m_Z) = 2.83 \ln(\alpha_{\text{GUT}}/\alpha_3(m_Z)) = 6.05. \quad (30)$$

The resulting correction to the  $\alpha_3$  prediction is

$$\Delta_\kappa \left( \frac{2\pi}{\alpha_3} \right) = \frac{9}{7} \ln Z_H^Y(m_Z) = 7.78, \quad (31)$$

which is just sufficient to compensate the negative two-loop effects in the last line of Table 1. Of course, at this stage our promising results should be taken with caution. Specifically, when talking about a strongly-coupled unified theory, one faces the danger of potentially large and incalculable corrections at the GUT scale which could, in principle, render the entire two-loop analysis obsolete. We postpone the discussion of these issues to Sect. 4. We also note that the influence of extra Yukawas *above* the GUT scale on the unified coupling has recently been discussed in [30].

---

<sup>5</sup> One writes  $\alpha_i^{-1}(\mu) = \mathcal{O}(1) + b'_i \ln \left( \frac{M_{\text{GUT}}}{\mu} \right)$  and neglects the  $\mathcal{O}(1)$  term.

### 3.3 Effect of extra Yukawas in models with perturbative gauge couplings

In this section we extend our previous results to a more general setting by lifting some of the simplifying ad hoc assumptions that were made so far. This means in particular that we introduce an explicit decoupling scale  $M$  for the extra matter fields. Also, from now on the messenger index  $n$  is allowed to attain only integer values (however, we only allow for  $n \geq 3$  since we assume at least one  $\mathbf{10} + \overline{\mathbf{10}}$  pair). The superpotential in the extended Yukawa sector is still specified by Eq. (18). In order to improve the accuracy of our predictions we will also take into account the contributions from the MSSM top Yukawa coupling. Since this coupling enters the one-loop RGEs of  $\alpha_\kappa$  and  $\alpha_{\overline{\kappa}}$  in a different manner, this step explicitly breaks the symmetry of our model with respect to  $\kappa$  and  $\overline{\kappa}$ . In particular, from now on we will distinguish explicitly between these two couplings.

The most important constraint which we impose on the models in this section is perturbativity of the gauge couplings. Among other things this implies that the analytical approach we developed in the previous section is no longer applicable. Instead we will employ an alternative technique, which will allow us to handle our models semi-analytically.

To this end we will have to deal with the two different regimes of the theory (the high-energy regime above  $M$  and the low-energy regime below  $M$ ) separately. Let us focus on the high-energy theory first. As was mentioned before, any MSSM field obeys a factorization property analogous to Eq. (20). This means that the gauge and Yukawa contributions to the matter field  $Z$ -factors decouple and can be analyzed independently. The gauge parts of the  $Z$ -factors are obtained from Eq. (17) by simply replacing  $m_Z \rightarrow M$ . In order to get the Yukawa parts we will need the one-loop RGEs for the  $Z^Y$  (above  $M$ ):

$$\begin{aligned} 2\pi \frac{d \ln Z_{H_U}^Y}{dt} &= -3\alpha_t - 3\alpha_\kappa, & 2\pi \frac{d \ln Z_{H_D}^Y}{dt} &= -3\alpha_{\overline{\kappa}}, \\ 2\pi \frac{d \ln Z_{U_3}^Y}{dt} &= -2\alpha_t, & 2\pi \frac{d \ln Z_{Q_3}^Y}{dt} &= -\alpha_t. \end{aligned} \quad (32)$$

Since we regard only the  $\kappa$ ,  $\overline{\kappa}$  and  $y_t$  Yukawas as non-vanishing, all other  $Z^Y$ -factors are irrelevant for our analysis. In the following we will also need the one-loop equations for the three aforementioned Yukawa couplings:

$$\begin{aligned} 2\pi \frac{d \ln \alpha_t}{dt} &= 6\alpha_t + 3\alpha_\kappa - \frac{13}{15}\alpha_1 - 3\alpha_2 - \frac{16}{3}\alpha_3 \\ 2\pi \frac{d \ln \alpha_\kappa}{dt} &= 6\alpha_\kappa + 3\alpha_t - \frac{13}{15}\alpha_1 - 3\alpha_2 - \frac{16}{3}\alpha_3 \\ 2\pi \frac{d \ln \alpha_{\overline{\kappa}}}{dt} &= 6\alpha_{\overline{\kappa}} - \frac{13}{15}\alpha_1 - 3\alpha_2 - \frac{16}{3}\alpha_3 \end{aligned} \quad (33)$$

Note that these identities generalize Eq. (22) to the case of non-vanishing  $\alpha_t$ . Following [6] we can combine Eqs. (33) and (32) to get a closed analytic expression for the Yukawa

parts of the relevant  $Z$ -factors:

$$\begin{aligned}
Z_{Q_3}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{1}{9} \frac{13}{15 b'_1}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{1}{9} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{1}{9} \frac{16}{3 b'_3}} \left( \frac{\alpha_{t,\text{GUT}}}{\alpha_t(M)} \right)^{\frac{2}{9}} \left( \frac{\alpha_{\kappa,\text{GUT}}}{\alpha_\kappa(M)} \right)^{-\frac{1}{9}} \\
Z_{U_3}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{2}{9} \frac{13}{15 b'_1}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{2}{9} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{2}{9} \frac{16}{3 b'_3}} \left( \frac{\alpha_{t,\text{GUT}}}{\alpha_t(M)} \right)^{\frac{4}{9}} \left( \frac{\alpha_{\kappa,\text{GUT}}}{\alpha_\kappa(M)} \right)^{-\frac{2}{9}} \\
Z_{H_D}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{1}{2} \frac{13}{15 b'_1}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{1}{2} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{1}{2} \frac{16}{3 b'_3}} \left( \frac{\alpha_{\bar{\kappa},\text{GUT}}}{\alpha_{\bar{\kappa}}(M)} \right)^{\frac{1}{2}} \\
Z_{H_U}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{2}{3} \frac{13}{15 b'_1}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{2}{3} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{2}{3} \frac{16}{3 b'_3}} \left( \frac{\alpha_{t,\text{GUT}}}{\alpha_t(M)} \right)^{\frac{1}{3}} \left( \frac{\alpha_{\kappa,\text{GUT}}}{\alpha_\kappa(M)} \right)^{\frac{1}{3}}.
\end{aligned} \tag{34}$$

The theory below the scale  $M$  is the MSSM. Therefore we can apply our formulas from Sect. 2.3 and App. B. The only modification is that we now integrate from  $m_Z$  to  $M$  rather than to  $M_{\text{GUT}}$ . As an illustrative example we record the result for the Yukawa part of the  $Z_{H_U}$  factor:

$$\frac{Z_{H_U}^Y(m_Z)}{Z_{H_U}^Y(M)} = \left( \frac{\alpha_1(M)}{\alpha_1(m_Z)} \right)^{\frac{13}{30 b_1}} \left( \frac{\alpha_2(M)}{\alpha_2(m_Z)} \right)^{\frac{3}{2 b_2}} \left( \frac{\alpha_3(M)}{\alpha_3(m_Z)} \right)^{\frac{8}{3 b_3}} \left( \frac{\alpha_t(M)}{\alpha_t(m_Z)} \right)^{\frac{1}{2}} \tag{35}$$

The calculation of the other  $Z$ -factors proceeds in a similar manner. The low-scale values  $Z(m_Z)$  are then obtained by multiplying the expressions for  $Z(M)$  and  $Z(m_Z)/Z(M)$ .

The brackets in Eqs. (34) and (35) involving gauge couplings can be evaluated analytically (by using the respective one-loop values). To calculate the Yukawa brackets we have solved the one-loop RGEs in Eq. (33) numerically by evolving them from the GUT down to the electroweak scale and using a  $\theta$ -function approximation at the decoupling scale  $M$ . The resulting low-scale values for the three Yukawa couplings were then substituted in Eqs. (34) and (35). In Tables 2 and 3 we have listed the two-loop corrections to  $2\pi/\alpha_3$  for different values of the two parameters  $M$  and  $n$ . We have also tested the sensitivity of our results against variations of the initial values  $\alpha_\kappa(M_{\text{GUT}})$  and  $\alpha_{\bar{\kappa}}(M_{\text{GUT}})$  (we remark that the values  $\alpha_\kappa(M_{\text{GUT}}) = \alpha_{\bar{\kappa}}(M_{\text{GUT}}) = 0.228, 0.457, 0.913$  correspond to  $1/6, 1/3, 2/3$  times the fixed point value 1.37 of the extra Yukawa couplings). In each case the initial value for the top Yukawa coupling at  $M_{\text{GUT}}$  has been adjusted to reproduce the correct low-energy parameter  $y_t(m_Z) = 0.99$  (see also App. B).

A quick glance at Tables 2 and 3 reveals that models with  $n = 4$  are favoured over their  $n = 5$  counterparts. The former are also of particular interest because of the low value of  $M$  [10–12, 31]. We have intentionally omitted the  $n = 3$  case because the  $n = 3$  models are unable to generate a sufficiently large GUT coupling (say  $\alpha_{\text{GUT}} \geq 0.2$ ).

It is important to note that the low-energy couplings  $\alpha_\kappa(m_Z)$  and  $\alpha_{\bar{\kappa}}(m_Z)$  are virtually insensitive to their input values at the GUT scale. This observation indicates a very straightforward way of increasing the two-loop prediction for  $2\pi/\alpha_3$  – namely by taking the input parameters  $\alpha_{\kappa,\text{GUT}}$  and  $\alpha_{\bar{\kappa},\text{GUT}}$  as large as possible. Note also that (in contradistinction to gauge couplings) the Yukawas do not exhibit any flavor enhancement

Table 2: Numerical results  $n = 4$  (one pair of extra Yukawa couplings)

$M(\text{GeV})$	$\alpha_{\text{GUT}}$	$\alpha_{\kappa,\text{GUT}} = \alpha_{\bar{\kappa},\text{GUT}}$	$\alpha_{\kappa}(M)$	$\alpha_{\bar{\kappa}}(M)$	$y_t(M_{\text{GUT}})$	$2\pi/\alpha_3$
<b>500</b>	<b>0.227</b>	<b>0.228</b>	<b>0.102</b>	<b>0.134</b>	<b>0.68</b>	<b>51.10</b>
500	0.227	<b>0.457</b>	<b>0.102</b>	<b>0.134</b>	<b>0.84</b>	<b>51.55</b>
500	0.227	<b>0.913</b>	<b>0.103</b>	<b>0.134</b>	<b>1.01</b>	<b>51.99</b>
500	0.227	<b>2.000</b>	<b>0.103</b>	<b>0.134</b>	<b>1.26</b>	<b>52.50</b>
500	0.227	<b>4.000</b>	<b>0.103</b>	<b>0.134</b>	<b>1.52</b>	<b>52.95</b>
500	0.227	<b>6.000</b>	<b>0.103</b>	<b>0.134</b>	<b>1.70</b>	<b>53.21</b>
<b>300</b>	<b>0.245</b>	<b>0.457</b>	<b>0.105</b>	<b>0.137</b>	<b>0.78</b>	<b>51.67</b>
300	0.245	<b>2.000</b>	<b>0.105</b>	<b>0.137</b>	<b>1.18</b>	<b>52.62</b>
300	0.245	<b>4.000</b>	<b>0.105</b>	<b>0.137</b>	<b>1.42</b>	<b>53.07</b>
300	0.245	<b>6.000</b>	<b>0.105</b>	<b>0.137</b>	<b>1.58</b>	<b>53.33</b>

 Table 3: Numerical results  $n = 5$  (one pair of extra Yukawa couplings)

$M(\text{GeV})$	$\alpha_{\text{GUT}}$	$\alpha_{\kappa,\text{GUT}} = \alpha_{\bar{\kappa},\text{GUT}}$	$\alpha_{\kappa}(M)$	$\alpha_{\bar{\kappa}}(M)$	$y_t(M_{\text{GUT}})$	$2\pi/\alpha_3$
<b><math>25 \cdot 10^4</math></b>	<b>0.229</b>	<b>0.228</b>	<b>0.097</b>	<b>0.121</b>	<b>0.70</b>	<b>50.34</b>
$25 \cdot 10^4$	0.229	<b>0.457</b>	<b>0.097</b>	<b>0.122</b>	<b>0.86</b>	<b>50.79</b>
$25 \cdot 10^4$	0.229	<b>0.913</b>	<b>0.098</b>	<b>0.122</b>	<b>1.04</b>	<b>51.24</b>
$25 \cdot 10^4$	0.229	<b>2.000</b>	<b>0.098</b>	<b>0.122</b>	<b>1.29</b>	<b>51.74</b>
$25 \cdot 10^4$	0.229	<b>4.000</b>	<b>0.098</b>	<b>0.122</b>	<b>1.56</b>	<b>52.19</b>
$25 \cdot 10^4$	0.229	<b>6.000</b>	<b>0.098</b>	<b>0.122</b>	<b>1.73</b>	<b>52.45</b>

(cf. Sect. 4). Therefore the actual expansion parameters are  $\alpha_{\kappa}/4\pi$  and  $\alpha_{\bar{\kappa}}/4\pi$ , which means that the strong coupling condition reads  $\alpha_{\kappa}/4\pi \sim \alpha_{\bar{\kappa}}/4\pi \sim 1$ . Following this line of thought we have considered models whose input values  $\alpha_{\kappa,\text{GUT}}$  and  $\alpha_{\bar{\kappa},\text{GUT}}$  are as high as 6.0.

We once again emphasize that the gauge couplings in this section are only allowed to attain perturbative values (in contradistinction to their Yukawa counterparts). As will be argued in Sect. 4 the lowering of the unified coupling from  $\alpha_{\text{GUT}} \sim \mathcal{O}(1)$  down to  $\sim 0.2$  can potentially have a dramatic impact on the magnitude and calculability of the high-scale threshold corrections around the GUT scale.

### 3.4 Models with further Yukawa couplings

In the following we will investigate the principal effect of introducing further non-standard Yukawa couplings. We focus on two extensions: First we consider adding new  $\mathbf{10} + \overline{\mathbf{10}}$  pairs which couple to the observable sector through an interaction analogous to Eq.(18). In particular we increase the messenger index to  $n \geq 6$ . Second, we explore a different possibility by restricting ourselves to the  $n = 4$  case but allowing for renormalizable interactions between the  $\mathbf{10} + \overline{\mathbf{10}}$  and the  $\mathbf{5} + \overline{\mathbf{5}}$  fields.

Following this line of thought let us now increase the number of additional vector-like

multiplets to  $n = 6$  by introducing two  $\mathbf{10} + \overline{\mathbf{10}}$  pairs. We postulate a superpotential of the form:

$$W \supset \kappa Q_e U_e H_U + \bar{\kappa} \bar{Q}_e \bar{U}_e H_D + \kappa' Q'_e U'_e H_U + \bar{\kappa}' \bar{Q}'_e \bar{U}'_e H_D \quad (36)$$

where the primed fields denote the matter content of the new  $\mathbf{10} + \overline{\mathbf{10}}$  pair. In the following we will treat the two Yukawa pairs  $(\kappa, \bar{\kappa})$  and  $(\kappa', \bar{\kappa}')$  on an equal footing, i.e. we assume that  $\kappa = \kappa'$  and  $\bar{\kappa} = \bar{\kappa}'$ . Note that by going from a superpotential of the form (18) to superpotential of the form (36) we only change the running of the *Yukawa* part of the  $Z$ -factors *above* the decoupling scale. The relevant one-loop RGEs are

$$\begin{aligned} 2\pi \frac{d \ln Z_{H_U}^Y}{dt} &= -3\alpha_t - 6\alpha_\kappa, & 2\pi \frac{d \ln Z_{H_D}^Y}{dt} &= -6\alpha_{\bar{\kappa}}, \\ 2\pi \frac{d \ln Z_{U_3}^Y}{dt} &= -2\alpha_t, & 2\pi \frac{d \ln Z_{Q_3}^Y}{dt} &= -\alpha_t \end{aligned} \quad (37)$$

for the  $Z^Y$  and

$$\begin{aligned} 2\pi \frac{d \ln \alpha_t}{dt} &= 6\alpha_t + 6\alpha_\kappa - \frac{13}{15}\alpha_1 - 3\alpha_2 - \frac{16}{3}\alpha_3 \\ 2\pi \frac{d \ln \alpha_\kappa}{dt} &= 9\alpha_\kappa + 3\alpha_t - \frac{13}{15}\alpha_1 - 3\alpha_2 - \frac{16}{3}\alpha_3 \\ 2\pi \frac{d \ln \alpha_{\bar{\kappa}}}{dt} &= 9\alpha_{\bar{\kappa}} - \frac{13}{15}\alpha_1 - 3\alpha_2 - \frac{16}{3}\alpha_3 \end{aligned} \quad (38)$$

for the Yukawas.

Combining Eqs. (37) and (38) we arrive at the analogue of Eq. (34):

$$\begin{aligned} Z_{Q_3}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{1}{12} \frac{13}{15} b'_1} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{1}{12} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{1}{12} \frac{16}{3b'_3}} \left( \frac{\alpha_{t,\text{GUT}}}{\alpha_t(M)} \right)^{\frac{1}{4}} \left( \frac{\alpha_{\kappa,\text{GUT}}}{\alpha_\kappa(M)} \right)^{-\frac{1}{6}} \\ Z_{U_3}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{1}{6} \frac{13}{15} b'_1} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{1}{6} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{1}{6} \frac{16}{3b'_3}} \left( \frac{\alpha_{t,\text{GUT}}}{\alpha_t(M)} \right)^{\frac{1}{2}} \left( \frac{\alpha_{\kappa,\text{GUT}}}{\alpha_\kappa(M)} \right)^{-\frac{1}{3}} \\ Z_{H_D}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{2}{3} \frac{13}{15} b'_1} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{2}{3} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{2}{3} \frac{16}{3b'_3}} \left( \frac{\alpha_{\bar{\kappa},\text{GUT}}}{\alpha_{\bar{\kappa}}(M)} \right)^{\frac{2}{3}} \\ Z_{H_U}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{3}{4} \frac{13}{15} b'_1} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{3}{4} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{3}{4} \frac{16}{3b'_3}} \left( \frac{\alpha_{t,\text{GUT}}}{\alpha_t(M)} \right)^{\frac{1}{4}} \left( \frac{\alpha_{\kappa,\text{GUT}}}{\alpha_\kappa(M)} \right)^{\frac{1}{2}}. \end{aligned} \quad (39)$$

The results from our numerical analysis are listed in Table 4. The prediction for  $\alpha_3$  improves only slightly in comparison to the ‘optimal’  $n = 4$  models with a single pair of extra Yukawas.

The next type of models we consider contain one pair of  $\mathbf{10} + \overline{\mathbf{10}}$  and one pair of  $\mathbf{5} + \overline{\mathbf{5}}$  extra multiplets. Using the standard decomposition  $\mathbf{5} = (\overline{D}_e, \overline{L}_e)$  and  $\overline{\mathbf{5}} = (D_e, L_e)$  we introduce a superpotential of the form:

$$W \supset \kappa Q_e U_e H_U + \bar{\kappa} \bar{Q}_e \bar{U}_e H_D + \lambda Q_e D_e H_D + \bar{\lambda} \bar{Q}_e \overline{D}_e H_U \quad (40)$$

Table 4: Numerical results  $n=6$  (two pairs of extra Yukawa couplings)

$M(\text{GeV})$	$\alpha_{\text{GUT}}$	$\alpha_{\kappa, \text{GUT}} = \alpha_{\bar{\kappa}, \text{GUT}}$	$\alpha_{\kappa}(M)$	$\alpha_{\bar{\kappa}}(M)$	$y_t(M_{\text{GUT}})$	$2\pi/\alpha_3$
$17 \cdot 10^6$	<b>0.227</b>	<b>0.228</b>	<b>0.064</b>	<b>0.080</b>	<b>1.26</b>	<b>51.08</b>
$17 \cdot 10^6$	0.227	<b>0.457</b>	<b>0.065</b>	<b>0.080</b>	<b>1.68</b>	<b>51.66</b>
$17 \cdot 10^6$	0.227	<b>0.913</b>	<b>0.065</b>	<b>0.081</b>	<b>2.24</b>	<b>52.25</b>
$17 \cdot 10^6$	0.227	<b>2.000</b>	<b>0.065</b>	<b>0.081</b>	<b>3.08</b>	<b>52.92</b>
$17 \cdot 10^6$	0.227	<b>4.000</b>	<b>0.065</b>	<b>0.081</b>	<b>4.06</b>	<b>53.52</b>
$17 \cdot 10^6$	0.227	<b>6.000</b>	<b>0.065</b>	<b>0.081</b>	<b>4.77</b>	<b>53.86</b>

Note that this is a direct extension of the  $n = 4$  models from Sect. 3.3 – we have simply added two new interactions to our superpotential. The calculation of the two-loop  $\alpha_3$ -correction proceeds exactly as before. Here we only list the relevant one-loop RGEs above the decoupling scale  $M$ :

$$\begin{aligned}
2\pi \frac{d \ln Z_{H_U}^Y}{dt} &= -3\alpha_t - 3\alpha_{\kappa} - 3\alpha_{\bar{\lambda}}, & 2\pi \frac{d \ln Z_{H_D}^Y}{dt} &= -3\alpha_{\bar{\kappa}} - 3\alpha_{\lambda}, \\
2\pi \frac{d \ln Z_{U_3}^Y}{dt} &= -2\alpha_t, & 2\pi \frac{d \ln Z_{Q_3}^Y}{dt} &= -\alpha_t
\end{aligned} \tag{41}$$

for the matter field  $Z^Y$ -factors and

$$\begin{aligned}
2\pi \frac{d \ln \alpha_t}{dt} &= 6\alpha_t + 3\alpha_{\kappa} + 3\alpha_{\bar{\lambda}} - \frac{13}{15}\alpha_1 - 3\alpha_2 - \frac{16}{3}\alpha_3 \\
2\pi \frac{d \ln \alpha_{\kappa}}{dt} &= 6\alpha_{\kappa} + 3\alpha_t + 3\alpha_{\bar{\lambda}} - \frac{13}{15}\alpha_1 - 3\alpha_2 - \frac{16}{3}\alpha_3 \\
2\pi \frac{d \ln \alpha_{\bar{\kappa}}}{dt} &= 6\alpha_{\bar{\kappa}} + 3\alpha_{\lambda} - \frac{13}{15}\alpha_1 - 3\alpha_2 - \frac{16}{3}\alpha_3 \\
2\pi \frac{d \ln \alpha_{\lambda}}{dt} &= 6\alpha_{\lambda} + 3\alpha_{\bar{\kappa}} - \frac{7}{15}\alpha_1 - 3\alpha_2 - \frac{16}{3}\alpha_3 \\
2\pi \frac{d \ln \alpha_{\bar{\lambda}}}{dt} &= 6\alpha_{\bar{\lambda}} + 3\alpha_t + 3\alpha_{\kappa} - \frac{7}{15}\alpha_1 - 3\alpha_2 - \frac{16}{3}\alpha_3
\end{aligned} \tag{42}$$

for the Yukawa couplings. As before we have defined  $\alpha_{\lambda} = \lambda^2/(4\pi)$  and  $\alpha_{\bar{\lambda}} = \bar{\lambda}^2/(4\pi)$ . The modification of Eq.(34) reads

$$\begin{aligned}
Z_{Q_3}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{1}{12} \frac{19}{15} b'_1} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{1}{12} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{1}{12} \frac{16}{3 b'_3}} \left( \frac{\alpha_{t, \text{GUT}}}{\alpha_t(M)} \right)^{\frac{1}{4}} \times \\
&\quad \times \left( \frac{\alpha_{\kappa, \text{GUT}}}{\alpha_{\kappa}(M)} \right)^{-\frac{1}{12}} \left( \frac{\alpha_{\bar{\lambda}, \text{GUT}}}{\alpha_{\bar{\lambda}}(M)} \right)^{-\frac{1}{12}} \\
Z_{U_3}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{1}{6} \frac{19}{15} b'_1} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{1}{6} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{1}{6} \frac{16}{3 b'_3}} \left( \frac{\alpha_{t, \text{GUT}}}{\alpha_t(M)} \right)^{\frac{1}{2}} \times \\
&\quad \times \left( \frac{\alpha_{\kappa, \text{GUT}}}{\alpha_{\kappa}(M)} \right)^{-\frac{1}{6}} \left( \frac{\alpha_{\bar{\lambda}, \text{GUT}}}{\alpha_{\bar{\lambda}}(M)} \right)^{-\frac{1}{6}}.
\end{aligned} \tag{43}$$

$$\begin{aligned}
Z_{H_D}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{1}{3} \frac{20}{15b'_1}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{2}{3} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{2}{3} \frac{16}{3b'_3}} \left( \frac{\alpha_{\bar{\kappa},\text{GUT}}}{\alpha_{\bar{\kappa}}(M)} \right)^{\frac{1}{3}} \left( \frac{\alpha_{\lambda,\text{GUT}}}{\alpha_{\lambda}(M)} \right)^{\frac{1}{3}} \\
Z_{H_U}^Y(M) &= \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_1(M)} \right)^{\frac{1}{4} \frac{33}{15b'_1}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_2(M)} \right)^{\frac{3}{4} \frac{3}{b'_2}} \left( \frac{\hat{\alpha}_{\text{GUT}}}{\alpha_3(M)} \right)^{\frac{3}{4} \frac{16}{3b'_3}} \left( \frac{\alpha_{t,\text{GUT}}}{\alpha_t(M)} \right)^{\frac{1}{4}} \times \\
&\quad \times \left( \frac{\alpha_{\kappa,\text{GUT}}}{\alpha_{\kappa}(M)} \right)^{\frac{1}{4}} \left( \frac{\alpha_{\bar{\lambda},\text{GUT}}}{\alpha_{\bar{\lambda}}(M)} \right)^{\frac{1}{4}}.
\end{aligned}$$

We have presented the results from our numerical analysis in Table 5. It is important to note that, already for moderate input values of the extra Yukawas at the GUT scale, we reach the region of the experimentally measured  $\alpha_3$ . Hence in models of this type it is no longer necessary to invoke excessively large Yukawa couplings at the high-scale. For this reason we have restricted ourselves to a region of parameter space where  $\alpha_{\kappa,\text{GUT}}, \alpha_{\bar{\kappa},\text{GUT}}, \alpha_{\lambda,\text{GUT}}, \alpha_{\bar{\lambda},\text{GUT}} < 1$ .

Table 5: Numerical results n=4 (two pairs of extra Yukawa couplings)

$M(\text{GeV})$	$\alpha_{\text{GUT}}$	$\alpha_{\kappa,\text{GUT}} = \alpha_{\bar{\kappa},\text{GUT}} = \alpha_{\lambda,\text{GUT}} = \alpha_{\bar{\lambda},\text{GUT}}$	$y_t(M_{\text{GUT}})$	$2\pi/\alpha_3$
<b>1000</b>	<b>0.206</b>	<b>0.228</b>	<b>2.25</b>	<b>52.39</b>
1000	0.206	<b>0.457</b>	<b>3.38</b>	<b>52.99</b>
1000	0.206	<b>0.913</b>	<b>5.20</b>	<b>53.58</b>

## 4 Threshold corrections and higher-order effects

In this section we discuss issues related to low- and high-energy thresholds and other higher-order corrections. The effect of the superpartner spectrum on the value of the strong coupling has been studied in detail in [32–34]. The analysis reveals that the low-energy thresholds potentially shift the predicted value of  $\alpha_3$  by a significant amount. For instance, it is well-known that certain SUSY spectra with light gluinos can compensate for the detrimental two-loop effect discussed in Sect. 2.3 and therefore bring the  $\alpha_3$ -prediction in line with the experimental value (see e.g. [35]). However, gluinos tend to be heavy in the simplest mediation scenarios and it generally requires a compensation of several mediation effects to make them light. A detailed study of concrete models realizing this possibility has recently appeared in [5].

Another potentially important contribution arises from heavy particle thresholds. For example, from Eqs. (5) or (8) it is clear that in order to shift the prediction for  $2\pi/\alpha_3$  by several units the logarithms of the mass ratios  $m_{X,Y}^h/m_3^h$  or  $m_{X,Y}^p/m_3^p$  have to be several units themselves. In other words, the Higgs triplets have to be  $\sim 10^2$  lighter than  $M_{\text{GUT}}$  and proton decay has to be avoided through some version of the missing partner mechanism (see [36] for references). Of course, thresholds with larger numerical



prefactors (and hence smaller required mass ratios) can arise in models with large GUT-scale representations (see, e.g., [32, 37–39]). A similar enhancement can come from large multiplicities of heavy states (this has in particular been argued in the context of certain string-motivated models [40]).<sup>6</sup> All of this clearly makes ‘GUT-scale thresholds’ a viable explanation of the precise value of  $\alpha_3$ .

The models considered in this paper offer an alternative solution to the two-loop  $\alpha_3$ -discrepancy. This solution differs conceptually from the aforementioned approaches as it does not rely on any type of threshold effects. It realizes a lower  $\alpha_3$  value at the expense of relatively large extra Yukawa couplings. The latter do not require any additional SU(5)-breaking effect (beyond the doublet-triplet splitting, which is anyway present in the MSSM). Nevertheless, precision at the high scale remains a critical issue and the rest of this section is devoted to its analysis.

The extra multiplets supporting the extra Yukawas raise the value of the GUT-scale gauge coupling. It is then tempting to consider the extreme case of such scenarios: Grand unification at the strong-coupling point. Even more conservatively, one might want to drop the GUT-assumption altogether and to demand only that all three SM gauge factors become strongly coupled at the same energy scale [14] (for early related work see [41]). The resulting high-scale error for the  $\alpha_3$  prediction can be estimated using the familiar one-loop formula

$$\frac{4\pi}{\alpha_i(m_Z)} = \frac{4\pi}{\alpha_i(M_{\text{GUT}})} + 2b_i \ln(M_{\text{GUT}}/m_Z). \quad (44)$$

Naively, ‘strong coupling’ means that the loop-expansion parameter  $g^2/(16\pi^2)$  is of order one. This would imply that  $4\pi/\alpha_i(M_{\text{GUT}}) = 1 \pm \mathcal{O}(1)$  in Eq. (44). The resulting error of the  $\alpha_3$  prediction is rather small, around 1%. However, because of the large number of flavors, the actual expansion parameter is in fact closer to  $\alpha$  rather than  $\alpha/(4\pi)$ .<sup>7</sup> Being at strong coupling then means that  $1/\alpha_i(M_{\text{GUT}}) = 1 \pm \mathcal{O}(1)$ , which corresponds to a  $\sim 10\%$  error of the  $\alpha_3$  prediction. While this easily brings the 2-loop MSSM prediction for  $\alpha_3$  in line with the data, it also makes any discussion of 2-loop effects obsolete: The error is simply too large.

For the purpose of the present paper, we adopt a different point of view: We assume that a model with true, calculable unification exists in principle and that the coupling strength is controlled by some high-scale holomorphic parameter (e.g. a string theory modulus). When talking about strongly-coupled unification, we assume that this model is realized in a region of its parameter space where  $\alpha_{\text{GUT}} \simeq \mathcal{O}(1)$ .

---

<sup>6</sup>For explicit orbifold constructions where the relevant corrections decouple from the string scale and their size could be checked straightforwardly see e.g. [42].

<sup>7</sup>There are at least two ways to see this: First, we focus on the contribution of all ‘flavors’ to the one-loop  $\beta$ -function of the QCD coupling at high scales. The corresponding  $\beta$ -function coefficient is  $2b_3^{\text{flavor}} = 2(n+6) \approx 20$  ( $n = 4$  or  $5$ ). This more than compensates for the suppression by  $4\pi$ , leaving us with a number close to  $\alpha$  as the actual expansion parameter. Alternatively, we consider the one-loop contribution of the gluons/gluinos,  $2b_3^{\text{color}} = 18$ . Once again, this is more than sufficient to cancel the factor of  $4\pi$ .

To be more specific, consider the string-theoretic (heterotic) formula [43]

$$f_i(S, T) = k_i S + \Delta_i(T) \quad (45)$$

for the gauge-kinetic functions of the three SM gauge groups. For the conventional embedding of  $G_{SM}$  in  $SU(5)$  and  $SU(5)$  in  $E_8$ , the  $k_i$  are unity, corresponding to tree-level unification. The crucial point is that the modulus governing the tree-level coupling strength (in this case the dilaton superfield  $S$ ) does not appear in the loop corrections. They depend on a set of different moduli which we collectively denote by  $T$ . The reason for this is basically the same as in field-theoretic arguments for 1-loop running: holomorphicity and shift symmetry of  $\text{Im}(S)$  [22, 24]. Now, moving the modulus  $S$  from its perturbative value to the region where the expansion parameter is  $\mathcal{O}(1)$ , we see that the high-scale non-universal correction  $\Delta_i(T)$  is not enhanced.<sup>8</sup> The only potential danger comes from non-perturbative extra terms  $\sim C_i \exp(-a_i S)$  which can appear on the r.h. side of Eq. (45). To keep such terms under control, we only have to assume that  $\exp(-\text{Re}S) \ll 1$  – a much weaker requirement than  $1/\text{Re}S \ll 1$ .

A further important issue is the error which builds up along the RG trajectory from  $m_Z$  to  $M_{\text{GUT}}$  as a result of using the two-loop instead of the full  $\beta$ -function for the gauge couplings. According to the previously discussed master formulae (cf. Eqs. (3) and (7)) this error is identical to the error of the  $\ln Z$  terms. At one loop (and focussing only on the gauge sector for simplicity) we have  $d \ln Z / dt \sim \alpha$ , which upon integration gives  $\ln Z \sim \ln t$ . It is clear that this leading order effect receives contributions from the entire integration range. We will now argue that higher-order corrections are UV dominated. To this end we recall that at two loops the RGE for a generic  $Z$ -factor has the form:

$$\frac{d \ln Z}{dt} \sim \alpha + \alpha^2, \quad (46)$$

where  $\alpha$  is the canonical (or physical) gauge coupling. From the anomaly relation Eq. (2) we obtain schematically

$$\alpha \sim \alpha_h + \alpha_h^2 \ln \alpha_h + \alpha_h^2 \ln Z \sim \frac{1}{t} + \frac{\ln t}{t^2}, \quad (47)$$

where we have only displayed the leading corrections. Note that we have used the (perturbatively exact) one-loop holomorphic gauge coupling  $\alpha_h \sim 1/t$  as well as the fact that (at one loop)  $\ln Z \sim \ln t$ . Substituting the above relation in the two-loop RGE (46) gives

$$\ln Z \sim \int \frac{dt}{t} + \int \frac{\ln t}{t^2} dt, \quad (48)$$

where terms  $\sim 1/t^3$  or higher were neglected. The first term on the r.h. side gives the previously discussed leading order  $\ln t$  effect. The second integral is UV dominated,

---

<sup>8</sup> The dual situation, where the leading-order gauge coupling is governed by the GUT-brane volume  $T$  and corrections (related to a higher-dimension operator) depend on the dilaton  $S$ , arises in F-theory GUTs. These corrections tend to aggravate the two-loop discrepancy for  $\alpha_3$  [44], potentially making ‘our’ Yukawa effect the more interesting (see also [45]).

i.e., this subleading effect can indeed be neglected at our level of accuracy. To be more precise, the corresponding correction is not enhanced by the parametrically large quantity  $t \sim \ln(M_{\text{GUT}}/m_Z)$  or a log thereof. Hence, it corresponds to an  $\mathcal{O}(1)$  correction to  $\ln Z$ . This is equivalent to a multiplicative  $\mathcal{O}(1)$  uncertainty of the high-scale  $Z$  factor, which we anyway have to accept in the absence of an explicit GUT model. The same argument goes through for contributions of even higher order.

To summarize, our strongly-coupled unification scenario is defined as follows: The unified gauge coupling is taken to be relatively large,  $\alpha_{\text{GUT}} \sim \mathcal{O}(1)$ , while non-perturbative corrections are still under control. This may be the case because there is at least a (small) hierarchy of the type  $\exp(-4\pi/\alpha_{\text{GUT}}) \ll 1$  or because the coefficients of such non-perturbative terms happen to be small. In such a setting, our 2-loop analysis of the strongly coupled model from Sect. 3.2 is meaningful and necessary.

We also emphasize that the aforementioned problems related to large and potentially incalculable GUT-scale corrections are automatically avoided in the models we considered in Sects. 3.3 and 3.4: All three gauge couplings remain within the perturbative domain throughout the entire energy range from  $m_Z$  to  $M_{\text{GUT}}$ .

## 5 Conclusions

In this paper we have shown that models with extra Yukawa couplings have a dramatic impact on the prediction for  $\alpha_3$  at the weak scale. They can bring the two-loop prediction in line with experimental data without appealing to large GUT-scale or weak-scale threshold corrections. This is an effect which has no analogue in the realm of MSSM physics – even the top Yukawa coupling is negligible in this context.

We introduced our main ideas using a simple model with strong GUT coupling. This model contains an extra  $\mathbf{10} + \overline{\mathbf{10}}$  pair with top-like Yukawa coupling to the MSSM Higgs doublets, together with further  $\mathbf{5} + \overline{\mathbf{5}}$  pairs making the GUT-scale gauge coupling strong. In this context, the main features and implications of our construction can be understood in a completely analytical manner.

We then demonstrated that our promising initial results retain their validity in situations where the gauge couplings do not leave the perturbative domain. This more complete and partially numerical analysis revealed, among other things, that large input values for the extra Yukawa couplings at the GUT scale can lead to an almost perfect prediction for the electroweak-scale strong coupling.

We also tested the sensitivity of our results to extensions of the minimal setting described above. In particular, the positive effect of the extra Yukawas on the unification prediction for  $\alpha_3$  is significantly further enhanced for a complete vector-like extra generation with both up-type and down-type Yukawa couplings. In such models, perfect agreement with experiment is achieved without invoking excessively large values for the extra Yukawa or gauge couplings at the GUT scale. By contrast, introducing several copies of  $\mathbf{10} + \overline{\mathbf{10}}$  with corresponding Yukawa couplings leads to a weaker enhancement

of the Yukawa effect. This can be traced to the increased scale  $M$  at which the extra multiplets decouple. Such a raised decoupling scale, which is necessary to keep the GUT coupling moderate, partially compensates for the positive effect of further extra Yukawas.

The presence of additional vector-like matter (at comparatively low-energies) and of extra Yukawa couplings, which have large values in the ultraviolet, can clearly have a significant impact on the MSSM phenomenology. We expect that the enhanced Higgs  $Z$  factors, which are at the heart of the effect we analyse, would in particular affect the values of the Higgs mass parameters  $m_{H_U}^2$ ,  $m_{H_D}^2$ ,  $\mu^2$  and  $B\mu$  which are expected in any concrete SUSY breaking model. We note that effects on the Higgs mass bounds in related non-supersymmetric models have recently been analysed in [46].

Finally, as we have discussed in some detail in our paper, the increased value of the GUT gauge coupling does not lead to a precision loss of the unification prediction for  $\alpha_3$ . The basic reason is that holomorphicity forbids the dominant higher-loop effects, which arise only via the non-holomorphic  $Z$  factors of MSSM chiral multiplets. However, non-perturbative GUT scale corrections clearly need to be controlled. This is easily possible in our setting, since it is not necessary to move to the actual strong-coupling point.

## Acknowledgments

We would like to thank Yasunori Nomura for many helpful and detailed discussions and for his hospitality during the visit of A.H. to the Berkeley Center for Theoretical Physics. We are also grateful to M. Luty, T. Plehn and M. Schmaltz for important discussions and comments.

## Appendix A: Gauge group $Z$ -factors in the MSSM

From the general formula (13) we obtain:

$$\begin{aligned} Z_{U(1)} &= \left( \frac{\alpha_{\text{GUT}}}{\alpha_1(m_Z)} \right)^{-\frac{2C_{F,1}}{b_1}} = \left( \frac{\alpha_{\text{GUT}}}{\alpha_1(m_Z)} \right)^{-\frac{1}{22}} = 0.96 \Rightarrow \ln Z_{U(1)} = -0.04 \\ Z_{SU(2)} &= \left( \frac{\alpha_{\text{GUT}}}{\alpha_2(m_Z)} \right)^{-\frac{2C_{F,2}}{b_2}} = \left( \frac{\alpha_{\text{GUT}}}{\alpha_2(m_Z)} \right)^{-\frac{3}{2}} = 0.75 \Rightarrow \ln Z_{SU(2)} = -0.29 \\ Z_{SU(3)} &= \left( \frac{\alpha_{\text{GUT}}}{\alpha_3(m_Z)} \right)^{-\frac{2C_{F,3}}{b_3}} = \left( \frac{\alpha_{\text{GUT}}}{\alpha_3(m_Z)} \right)^{\frac{8}{9}} = 0.39 \Rightarrow \ln Z_{SU(3)} = -0.93 . \end{aligned} \quad (49)$$

Here we have used a one-loop GUT coupling  $2\pi/\alpha_{\text{GUT}} = 2\pi \cdot 24.3 = 153$ . The quadratic Casimirs  $C_{F,i}$  as well as the one-loop MSSM  $\beta$ -function coefficients are given by:

$$2C_{F,i} = \left( \frac{3}{10}, \frac{3}{2}, \frac{8}{3} \right) \quad \text{and} \quad b_i = \left( \frac{33}{5}, 1, -3 \right) . \quad (50)$$

## Appendix B: Effect of top in the MSSM

In this appendix we analyze the effect of the top Yukawa coupling on the two-loop MSSM  $\alpha_3$ -prediction. We recall that in the presence of Yukawa couplings the matter field  $Z$ -factors factorize as  $Z = Z^G \times Z^Y$ , where, as before, the gauge part is given by Eqs. (12) and (13). In order to obtain the Yukawa sector contribution we start with the one-loop RGEs

$$2\pi \frac{d \ln Z_{H_U}^Y}{dt} = -3\alpha_t, \quad 2\pi \frac{d \ln Z_{U_3}^Y}{dt} = -2\alpha_t, \quad 2\pi \frac{d \ln Z_{Q_3}^Y}{dt} = -\alpha_t, \quad (51)$$

where we have defined  $\alpha_t = y_t^2/4\pi$ , with  $y_t$  being the *canonical* top Yukawa coupling. The subscript ‘3’ in ‘ $U_3$ ’ and ‘ $Q_3$ ’ refers to the third generation quark-antiquark particles. We also have the following one-loop equation for  $\alpha_t$ :

$$2\pi \frac{d \ln \alpha_t}{dt} = 6\alpha_t - \frac{16}{3}\alpha_3 - 3\alpha_2 - \frac{13}{15}\alpha_1. \quad (52)$$

Following [6], we combine Eqs. (51) and (52), finding

$$2\pi \frac{d \ln Z_{H_U}^Y}{dt} = -\frac{8}{3}\alpha_3 - \frac{3}{2}\alpha_2 - \frac{13}{30}\alpha_1 - 2\pi \frac{1}{2} \frac{d \ln \alpha_t}{dt}. \quad (53)$$

If we now express each of the three gauge couplings through their respective one-loop RGEs,

$$\frac{d \ln \alpha_i^{-1}}{dt} = -\frac{b_i}{2\pi} \alpha_i, \quad (54)$$

we can integrate Eq. (53) analytically:

$$Z_{H_U}^Y(m_Z) = \left( \frac{\alpha_{\text{GUT}}}{\alpha_1(m_Z)} \right)^{\frac{13}{30b_1}} \left( \frac{\alpha_{\text{GUT}}}{\alpha_2(m_Z)} \right)^{\frac{3}{2b_2}} \left( \frac{\alpha_{\text{GUT}}}{\alpha_3(m_Z)} \right)^{\frac{8}{3b_3}} \left( \frac{\alpha_t(M_{\text{GUT}})}{\alpha_t(m_Z)} \right)^{\frac{1}{2}}. \quad (55)$$

As already mentioned, we focus on the moderately large  $\tan \beta$  region, where  $m_t = y_t v$  with  $v = 174$  GeV. Using the low-scale values  $m_t = 173$  GeV and  $y_t(m_Z) = 0.99$ , we solve Eq. (52) numerically, using the explicit one-loop formulae for  $\alpha_i(t)$ . The resulting high-scale value  $y_t(M_{\text{GUT}}) = 0.57$  is then used to obtain  $\ln Z_{H_U}^Y(m_Z) = 0.74$ . Furthermore, Eqs. (51) imply  $Z_{U_3}^Y = (Z_{H_U}^Y)^{2/3}$  and  $Z_{Q_3}^Y = (Z_{H_U}^Y)^{1/3}$ . Thus, we find

$$\Delta_{\text{top}} \left( \frac{2\pi}{\alpha_3} \right) = \frac{3}{2} \ln Z_{Q_3}^Y - \frac{15}{14} \ln Z_{U_3}^Y + \frac{9}{14} \ln Z_{H_U}^Y = \frac{3}{7} \ln Z_{H_U}^Y = 0.32. \quad (56)$$

While, as is well known, this helps in lowering the  $\alpha_3$  prediction, the effect is far too small.

## Appendix C: Matter field Z-factors for $n \geq 1$

The numerical values of the wavefunction renormalization factors associated with the three gauge groups are listed in Table 6.

Using these results it is then easy to calculate the two-loop corrections to  $2\pi/\alpha_3$  arising from the different matter field  $Z$ -factors (cf. Table 7).

Table 6: Gauge group Z-factors

$n$	$\ln Z_{SU(3)}$	$\ln Z_{SU(2)}$	$\ln Z_{U(1)}$
1	- 1.06	- 0.33	- 0.04
2	- 1.27	- 0.38	- 0.05
3	- 1.63	- 0.46	- 0.06
4	- 2.53	- 0.66	- 0.08
4.45	- 3.87	- 0.93	- 0.11

Table 7: Matter field contributions

$n$	$\frac{27}{14} \ln Z_L$	$\frac{9}{2} \ln Z_Q$	$-\frac{9}{7} \ln Z_E$	$-\frac{45}{14} \ln Z_U$	$-\frac{27}{14} \ln Z_D$	$\frac{9}{14} \ln Z_{H_U} + \frac{9}{14} \ln Z_{H_D}$
1	-0.72	-6.28	0.21	3.64	2.07	-0.46
2	-0.83	-7.46	0.26	4.37	2.49	-0.54
3	-1.01	-9.44	0.27	5.64	3.19	-0.66
4	-1.42	-14.40	0.41	8.59	4.95	-0.94
4.45	-2.00	-21.67	0.57	13.07	7.56	-1.34

## References

- [1] S. Dimopoulos, S. Raby and F. Wilczek, Phys. Rev. D **24** (1981) 1681.
- [2] W. Buchmüller, K. Hamaguchi, O. Lebedev and M. Ratz, “Supersymmetric standard model from the heterotic string,” Phys. Rev. Lett. **96** (2006) 121602 [arXiv:hep-ph/0511035] and Nucl. Phys. B **785** (2007) 149 [arXiv:hep-th/0606187]; O. Lebedev, H. P. Nilles, S. Raby, S. Ramos-Sanchez, M. Ratz, P. K. S. Vaudrevange and A. Wingerter, “A mini-landscape of exact MSSM spectra in heterotic orbifolds,” Phys. Lett. B **645** (2007) 88 [arXiv:hep-th/0611095]; V. Braun, Y. H. He, B. A. Ovrut and T. Pantev, “The exact MSSM spectrum from string theory,” JHEP **0605** (2006) 043 [arXiv:hep-th/0512177].
- [3] C. Beasley, J. J. Heckman and C. Vafa, “GUTs and Exceptional Branes in F-theory” JHEP **0901** (2009) 058 [arXiv:0802.3391 [hep-th]] and JHEP **0901** (2009) 059 [arXiv:0806.0102 [hep-th]]; R. Donagi and M. Wijnholt, “Model Building with F-Theory,” arXiv:0802.2969 [hep-th].
- [4] C. Adam, J. L. Kneur, R. Lafaye, T. Plehn, M. Rauch and D. Zerwas, “Measuring Unification,” arXiv:1007.2190 [hep-ph].
- [5] S. Raby, M. Ratz and K. Schmidt-Hoberg, “Precision gauge unification in the MSSM,” arXiv:0911.4249 [hep-ph].
- [6] G. Amelino-Camelia, D. Ghilencea and G. G. Ross, “The effect of Yukawa couplings on unification predictions and the non-perturbative limit,” Nucl. Phys. B **528** (1998) 35 [arXiv:hep-ph/9804437].

- [7] T. Moroi and Y. Okada, “Radiative corrections to Higgs masses in the supersymmetric model with an extra family and antifamily,” *Mod. Phys. Lett. A* **7** (1992) 187 ;  
T. Moroi and Y. Okada, “Upper bound of the lightest neutral Higgs mass in extended supersymmetric Standard Models,” *Phys. Lett. B* **295** (1992) 73.
- [8] K. Tobe and J. D. Wells, “Higgs boson mass limits in perturbative unification theories,” *Phys. Rev. D* **66** (2002) 013010 [arXiv:hep-ph/0204196].
- [9] K. S. Babu, I. Gogoladze and C. Kolda, “Perturbative unification and Higgs boson mass bounds,” [arXiv:hep-ph/0410085].
- [10] K. S. Babu, I. Gogoladze, M. U. Rehman and Q. Shafi, “Higgs Boson Mass, Sparticle Spectrum and Little Hierarchy Problem in Extended MSSM,” *Phys. Rev. D* **78** (2008) 055017 [arXiv:0807.3055 [hep-ph]];  
S. P. Martin, “Extra vector-like matter and the lightest Higgs scalar boson mass in low-energy supersymmetry,” *Phys. Rev. D* **81** (2010) 035004 [arXiv:0910.2732 [hep-ph]].
- [11] P. W. Graham, A. Ismail, S. Rajendran and P. Saraswat, “A Little Solution to the Little Hierarchy Problem: A Vector-like Generation,” *Phys. Rev. D* **81** (2010) 055016 [arXiv:0910.3020 [hep-ph]].
- [12] S. P. Martin, “Raising the Higgs mass with Yukawa couplings for isotriplets in vector-like extensions of minimal supersymmetry,” arXiv:1006.4186 [hep-ph].
- [13] R. Barbieri, L. J. Hall, A. Y. Papaioannou, D. Pappadopulo and V. S. Rychkov, “An alternative NMSSM phenomenology with manifest perturbative unification,” *JHEP* **0803** (2008) 005 [arXiv:0712.2903 [hep-ph]].
- [14] T. Moroi, H. Murayama and T. Yanagida, “The Weinberg angle without grand unification,” *Phys. Rev. D* **48** (1993) 2995 [arXiv:hep-ph/9306268].
- [15] C. F. Kolda and J. March-Russell, “Low-energy signatures of semi-perturbative unification,” *Phys. Rev. D* **55** (1997) 4252 [arXiv:hep-ph/9609480].
- [16] D. Ghilencea, M. Lanzagorta and G. G. Ross, “Unification predictions,” *Nucl. Phys. B* **511** (1998) 3 [arXiv:hep-ph/9707401].
- [17] D. Ghilencea, M. Lanzagorta and G. G. Ross, “Strong unification,” *Phys. Lett. B* **415** (1997) 253 [arXiv:hep-ph/9707462].
- [18] J. L. Jones, “Gauge Coupling Unification in MSSM + 5 Flavors,” *Phys. Rev. D* **79** (2009) 075009 [arXiv:0812.2106 [hep-ph]].
- [19] J. Kopp, M. Lindner, V. Niro and T. E. J. Underwood, “On the Consistency of Perturbativity and Gauge Coupling Unification,” *Phys. Rev. D* **81** (2010) 025008 [arXiv:0909.2653 [hep-ph]].

- [20] R. Sato, T. T. Yanagida and K. Yonekura, “Relaxing a constraint on the number of messengers in a low-scale gauge mediation,” *Phys. Rev. D* **81** (2010) 045003 [arXiv:0910.3790 [hep-ph]].
- [21] S. Abel and V. V. Khoze, “Direct Mediation, Duality and Unification,” *JHEP* **0811** (2008) 024 [arXiv:0809.5262 [hep-ph]].
- [22] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, “Exact Gell-Mann-Low Function Of Supersymmetric Yang-Mills Theories From Instanton Calculus,” *Nucl. Phys. B* **229** (1983) 381;  
M. A. Shifman and A. I. Vainshtein, “Solution of the Anomaly Puzzle in SUSY Gauge Theories and the Wilson Operator Expansion,” *Nucl. Phys. B* **277** (1986) 456 and “On holomorphic dependence and infrared effects in supersymmetric gauge theories,” *Nucl. Phys. B* **359** (1991) 571.
- [23] M. A. Shifman, “Little Miracles of Supersymmetric Evolution of Gauge Couplings,” *Int. J. Mod. Phys. A* **11** (1996) 5761 [arXiv:hep-ph/9606281].
- [24] N. Arkani-Hamed and H. Murayama, “Holomorphy, rescaling anomalies and exact beta functions in supersymmetric gauge theories,” *JHEP* **0006** (2000) 030 [arXiv:hep-th/9707133] and “Renormalization group invariance of exact results in supersymmetric gauge theories,” *Phys. Rev. D* **57** (1998) 6638 [arXiv:hep-th/9705189].
- [25] I. Jack, D. R. T. Jones and C. G. North, “Scheme dependence and the NSVZ beta-function,” *Nucl. Phys. B* **486** (1997) 479 [arXiv:hep-ph/9609325];  
I. Jack, D. R. T. Jones and A. Pickering, “The connection between the DRED and NSVZ renormalisation schemes,” *Phys. Lett. B* **435** (1998) 61 [arXiv:hep-ph/9805482].
- [26] L. Calibbi, L. Ferretti, A. Romanino and R. Ziegler, “Gauge coupling unification, the GUT scale, and magic fields,” *Phys. Lett. B* **672** (2009) 152 [arXiv:0812.0342 [hep-ph]].
- [27] M. S. Chanowitz, “Bounding CKM Mixing with a Fourth Family,” *Phys. Rev. D* **79** (2009) 113008 [arXiv:0904.3570 [hep-ph]].
- [28] M. Bobrowski, A. Lenz, J. Riedl and J. Rohrwild, “How Much Space Is Left For A New Family Of Fermions?,” *Phys. Rev. D* **79** (2009) 113006 [arXiv:0902.4883 [hep-ph]].
- [29] B. Pendleton and G. G. Ross, “Mass And Mixing Angle Predictions From Infrared Fixed Points,” *Phys. Lett. B* **98** (1981) 291.
- [30] A. Aranda, J. L. Diaz-Cruz and A. D. Rojas, “Anomalies, Beta Functions and Supersymmetric Unification with Multi-Dimensional Higgs Representations,” *Phys. Rev. D* **80** (2009) 085027 [arXiv:0907.4552 [hep-ph]].



- [31] J. Jiang, T. Li and D. V. Nanopoulos, “Testable Flipped  $SU(5) \times U(1)$  Models,” Nucl. Phys. B **772** (2007) 49 [arXiv:hep-ph/0610054];  
C. Liu, “Supersymmetry and Vector-like Extra Generation,” Phys. Rev. D **80** (2009) 035004 [arXiv:0907.3011 [hep-ph]].
- [32] P. Langacker and N. Polonsky, “The Strong coupling, unification, and recent data,” Phys. Rev. D **52** (1995) 3081 [arXiv:hep-ph/9503214];  
P. Langacker and N. Polonsky, “Uncertainties in coupling constant unification,” Phys. Rev. D **47** (1993) 4028 [arXiv:hep-ph/9210235].
- [33] M. S. Carena, S. Pokorski and C. E. M. Wagner, “On the unification of couplings in the minimal supersymmetric Standard Model,” Nucl. Phys. B **406** (1993) 59 [arXiv:hep-ph/9303202].
- [34] J. Bagger, K. T. Matchev and D. Pierce, “Precision corrections to supersymmetric unification,” Phys. Lett. B **348** (1995) 443 [arXiv:hep-ph/9501277].
- [35] L. Roszkowski and M. A. Shifman, “Reconciling supersymmetric grand unification with  $\alpha^{-1} s(m(Z))$  approximates 0.11,” Phys. Rev. D **53** (1996) 404 [arXiv:hep-ph/9503358].
- [36] S. Raby, “Grand Unified Theories,” (published in C. Amsler *et al.* [Particle Data Group], Phys. Lett. B **667** (2008) 1).
- [37] D. Ring, S. Urano and R. L. Arnowitt, “Planck scale physics and the testability of  $SU(5)$  supergravity GUT,” Phys. Rev. D **52** (1995) 6623 [arXiv:hep-ph/9501247].
- [38] P. Nath and P. Fileviez Perez, “Proton stability in grand unified theories, in strings, and in branes,” Phys. Rept. **441** (2007) 191 [arXiv:hep-ph/0601023].
- [39] B. Dutta, Y. Mimura and R. N. Mohapatra, “Proton Decay and Flavor Violating Thresholds in  $SO(10)$  Models,” Phys. Rev. Lett. **100** (2008) 181801 [arXiv:0712.1206 [hep-ph]].
- [40] G. G. Ross, “Wilson line breaking and gauge coupling unification,” arXiv:hep-ph/0411057.
- [41] G. Parisi, “On The Value Of Fundamental Constants,” Phys. Rev. D **11** (1975) 909.
- [42] M. Blaszczyk, S. G. Nibbelink, M. Ratz, F. Ruehle, M. Trapletti and P. K. S. Vaudrevange, “A  $Z_2 \times Z_2$  standard model,” Phys. Lett. B **683** (2010) 340 [arXiv:0911.4905 [hep-th]];  
M. Blaszczyk, S. G. Nibbelink, F. Ruehle, M. Trapletti and P. K. S. Vaudrevange, “Heterotic MSSM on a Resolved Orbifold,” arXiv:1007.0203 [hep-th].
- [43] H. P. Nilles, “The Role Of Classical Symmetries In The Low-Energy Limit Of Superstring Theories” Phys. Lett. B **180** (1986) 240;  
L. J. Dixon, V. Kaplunovsky and J. Louis, “On Effective Field Theories Describing (2,2) Vacua of the Heterotic String,” Nucl. Phys. B **329** (1990) 27;

- I. Antoniadis, K. S. Narain and T. R. Taylor, “Higher Genus String Corrections To Gauge Couplings,” *Phys. Lett. B* **267** (1991) 37.
- [44] R. Donagi and M. Wijnholt, arXiv:0808.2223 [hep-th];  
 R. Blumenhagen, “Gauge Coupling Unification In F-Theory Grand Unified Theories,” *Phys. Rev. Lett.* **102** (2009) 071601 [arXiv:0812.0248 [hep-th]].
- [45] J. J. Heckman and C. Vafa, “An Exceptional Sector for F-theory GUTs,” arXiv:1006.5459 [hep-th];  
 E. Dudas and E. Palti, “On hypercharge flux and exotics in F-theory GUTs,” arXiv:1007.1297 [hep-th];  
 J. Jiang, T. Li, D. V. Nanopoulos and D. Xie, “Flipped SU(5) X U(1) Models from F-Theory,” *Nucl. Phys. B* **830** (2010) 195 [arXiv:0905.3394 [hep-th]];  
 T. Li, D. V. Nanopoulos and J. W. Walker, “Elements of F-theory Proton Decay,” arXiv:1003.2570 [hep-ph].
- [46] I. Gogoladze, B. He and Q. Shafi, “New Fermions at the LHC and Mass of the Higgs Boson,” arXiv:1004.4217 [hep-ph].